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THE VALIDITY OF THE RAYLEIGH EXPANSION

M. Neviere and M. Cadilhac*

SUMMARY. A new method to investigate the diffraction of an electromagnetic wave by a grating of infinite conductivity is given. This method, based on a convenient conformal mapping, leads to a general discussion of the validity of the Rayleigh expansion.

1. Formulation of the Problem /235**

We will refer the space to a rectangular triad (Oxyz). We will consider a periodic cylindrical surface having a period of 2π, whose generators are parallel to the Oz axis. The directrix located in the plane xOy has the equation $y = g(x)$ (Figure 1). The part of the space defined by $y \leq g(x)$ is filled with a metal having infinite conductivity. A plane electromagnetic wave having a wave vector k and linearly polarized impinges on the grating described in this way at an incidence angle of θ. We will attempt to determine the distribution of the energy diffracted among the different orders.

Because of the polarization under consideration [4], we will consider the unknown of the problem to be the component along the Oz axis of the electrical field vector ("case E //"), or the component along the Oz axis of the magnetic excitation vector ("case H //"). This unknown $f(x, y)$ is therefore a solution of problem (I) according to [1]:

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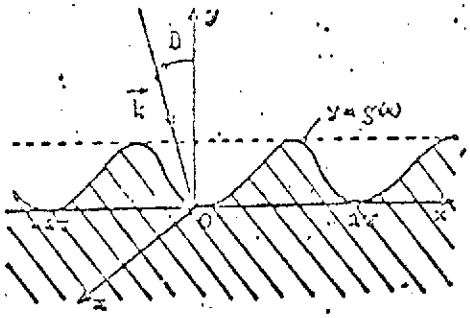

Figure 1. Schematic representation of a grating.

$$(\Delta + k^2) f(x,y) = 0 \,, \quad \text{for } y > g(x); \tag{1}$$

$$f(x+2\pi,\, y) = f(x,y)\, \exp[ik\, 2\pi \sin\theta] \tag{2}$$

("pseudo periodicity" property):

$(f - f_{incident})$ verifies an "outgoing wave condition" for

$$y \to +\infty; \tag{3}$$

$$\left.\begin{array}{l} f = 0 \text{ in the case E //} \\ \partial f/\partial n = 0 \text{ in the case H //} \end{array}\right\} \text{ for } y = g(x); \tag{4}$$

$\partial f/\partial n$ designates the normal derivative of the function $f(x, y)$ and k is the modulus of the wave vector of the incoming wave. I is the root of (-1) having the argument $+\frac{1}{2}\pi$.

2. Formulation of the Problem (I)

Condition (4) often makes it difficult to solve this problem. It would therefore be expedient to find a point transformation which would simplify its formulation. Such a transformation must maintain conditions (2) and (3) and must not overly complicate (1).

It is possible to verify that the family of conformal transformations defined by:

$$z = Z + \sum_{n=0}^{\infty} a_n \exp[inZ] \,, \tag{5}$$

where $z = x + iy$ and $Z = X + iY$,

— satisfies Conditions (2) and (3)

— replaces Equation (1) by

$$\frac{\partial^2 F(X,Y)}{\partial X^2} + \frac{\partial^2 F(X,Y)}{\partial Y^2} + K^2(X,Y)F(X,Y) = 0, \qquad (6)$$ /236

with $K^2(X,Y) = k^2|dz/dZ|^2$.

— according to the theorems for conformal transformations, it is possible to obtain the transformation of the directrix of a grating in the OX axis by a suitable choice of the α_n.

The new coefficient $K^2(X,Y)$ is then periodic and has the period 2π. In addition, it exponentially strives to k^2 when $Y \to \infty$. If we assume that for $Y > A$, where A is a number to be determined using some tests, $K^2(X,Y) = k^2$, we are led to a previously studied [2] limit problem.

Two types of problems can be considered:

(a) the grating profile is given, and the transformation coefficients α_n are determined which transform this profile into the OX axis.

(b) the collection of coefficients is given and the corresponding grating is studied.

In the two cases, we reach the problem discussed in [2].

3. Relationship Between the Rayleigh Expansion and the Conformal Transformation

The fundamental hypothesis of Rayleigh consists of assuming that above the grating the total field can be expanded into plane waves having the form

$$E(x,y) = \sum_{n=-\infty}^{+\infty} B_n \exp[i\chi_n y] \exp[i(n + k \sin\theta)x] + \exp[-iky] ,$$

where

$$\chi_n = [k^2 - (n + k \sin\theta)^2]^{1/2}, \text{ if } k^2 - (n + k\sin\theta)^2 > 0;$$
$$= i[(n + k\sin\theta)^2 - k^2]^{1/2}, \text{ if } k^2 - (n + k\sin\theta)^2 < 0.$$

If we refer to Millar [3], the solutions of the Helmholtz and Laplace equations will satisfy the same conditions at the limit and will have the same singularities. We can then study the corresponding "static" case.

The "static" Rayleigh expansion is then written as:

$$E(x,y) = \sum_{n=-\infty}^{+\infty} B_n \exp[-|n|y] \exp[inx] + y .$$

Let us consider the function $Z(z) = X(x, y) + iY(x, y)$. Y satisfies $\Delta Y = 0$, as an imaginary part of an analytic function; $Y(x, g(x)) = 0$ in terms of its construction.

Therefore $Y(x, y)$ is the solution of the electrostatic problem corresponding to (I) and can be identified with the static $E(x, y)$.

From this we can derive the fact that the validity of the static Rayleigh expansion is proven if we can show that:

$$Y(x,y) = \sum_{n=-\infty}^{+\infty} B_n \exp[-|n|y] \exp[inx] + y \,,$$

if $y \geq g(x)$.

It is easy to show that this implies that:

$$Z = z + \sum_{n=1}^{\infty} 2iB_n \exp[inz] \,,$$

which shows that the function Z(z) is holomorphic $\forall y \geq g(x)$.

We can therefore see that the investigation of the validity of the Rayleigh expansion leads to investigation of the singularities of Z(z).

We must now determine the possible singularities of Z(z), which is the inverse function of an analytic function in a finite domain. In effect, the transformation (5) means that there is a two-way correspondence between a point $M(x, y \geq g(x))$ and $M'(X, Y \geq 0)$. The possible singularities are therefore necessarily located in the domain limited by the curves having the equation y = 0 and y = g(x).

Let us limit ourselves to the case where the number of coefficients α_n entering in (5) is finite. The function z(Z) is therefore a whole function. Under these conditions, the only singularities possible are branching points which are obtained when dz/dZ = 0.

4. Applications

4.1 Case of Cycloidal Gratings

Let us recall the fact that it is this type of profile which can be used for the mathematical representation of holographic gratings [4]:

In (5), we will set $a_n = 0 \quad \forall n \neq 1$

$$\Rightarrow z = Z + a_1 \exp[iZ].$$

$$\Rightarrow z = Z + ia \exp[iZ].$$

We will set $a_1 = ia$ where a is real.

By setting Y = 0, we obtain the corresponding profile: /237

$$x = X - a \sin X; \qquad y = a \cos X,$$

which are parametric equations of a shortened cycloid (Figure 2).

Let us determine the region of validity of the Rayleigh expansion.

$$dz/dZ = 1 - a \exp[iZ] = 0,$$

if $Z = Z_0 = i \log a + 2p\pi$.

$\Rightarrow z_0 = i(1 + \log a) + 2p\pi$, where p is a whole number.

The field E(x, y) can therefore be represented in terms of plane waves only if $y > \mathrm{Im}(z_0)$. Consequently, if $\mathrm{Im}(z_0) > -a$, the Rayleigh expansion can only be used to translate the boundary conditions at the grating surface.

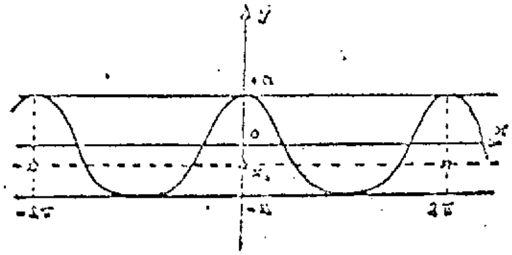

Figure 2. Region in which the Rayleigh expansion is valid for a cycloidal grating.

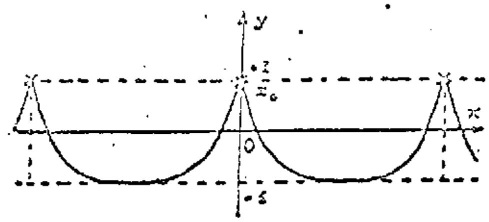

Figure 3. Localization of the singularities in the case a = 1.

The case is realistic if $a > a_0 \approx 0.2786$, which is a solution of the transcendental equation

$$1 + a + \log a = 0 . \tag{7}$$

The Rayleigh expansion cannot be valid for translating the boundary conditions over the entire grating surface except for sufficiently flat profiles, such that a will be smaller than α_0.

4.2 Special Case a = 1 (Figure 3)

$$z_0 = i(1 + \log a) = i + 2p\pi .$$

The singularities are located at the grating points, and the Rayleigh expansion is never valid.

4.3 Case of Small Scale Gratings

This type of profile requires an infinite number of coefficients α_n. It is difficult to determine the transcendental equation corresponding to (7). We can also reason differently:

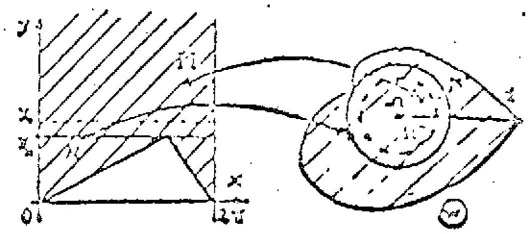

Figure 4. Correspondence between the $g = x + iy$ and the w plane.

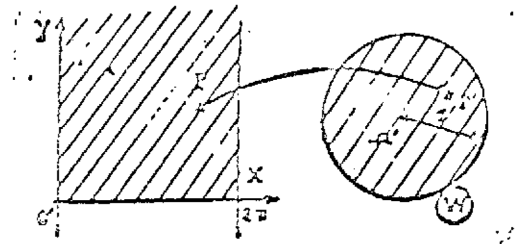

Figure 5. Correspondence between the $Z = X + iY$ and the W plane.

Let us set: $$w = \exp[iz]; \quad (8)$$

$$W = \exp[iZ]. \quad (9)$$

The relationships (8) and (9) respectively provide for the correspondences shown in Figures 4 and 5.

It is easy to verify that the transformation defined by (5) is equivalent to:

$$w(W) = \sum_{n=1}^{\infty} c_n W^n, \quad (10)$$

and that the Rayleigh expansion is valid if the function W(w) is holomorphic in the cross-hatched region of the plane (w), that is, if the expansion (10) converges. This convergence occurs within /238 a circle having its center at Ω (image at the point at infinity of the plane xOy), having a radius R_M equal to the distance from Ω to the closest singularity of w(W). We can therefore see that the Rayleigh expansion is only valid for $y < yM$. It is therefore not valid for a small scale grating,* and more generally, it is not valid for a grating the profile of which has angular points directed towards $y > 0$.

*This result is only valid for exceptional conditions, such as the grating of Marechel and Stroke [5].

5. Conclusion

The family of conformal transformations proposed therefore represents a simple and permanent method of numerically finding the region of validity of the Rayleigh expansion for gratings having a wide variety of profiles.

Other applications, for example sinusoidal profiles, are presently being studied.

REFERENCES

1. Petit, R. Rev. Opt., Vol. 6, 1966, p. 249; and Wirgin, A. Theses No. C. N. R. S. A. O., 1967, p, 1429.
2. Cerutti-Maori, G., R. Petit and M. Cadilhac. Compt. Rend. Acad. Sci., Vol. 268, 1969, Paris, p. 1060.
3. Millar, R. F. Proc. Cambridge Phil. Soc., to be published.
4. Maystre, D. and R. Petit. Opt. Commun., to be published.
5. Marechal, A. and G. W. Stroke, Compt. Rend. Acad. Sci., Vol. 249, 1959, Paris, p. 2042.

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